# Statistical Analysis of the Level Crossings and Duration of Fades of the Signal from an Energy Density Mobile Radio Antenna

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(Manuscript received September 30, 1966)

*A theoretical analysis of signal fading using an energy density antenna is developed and compared with that from an isotropic antenna. The energy density antenna provides a signal proportional to the energy density of the mobile radio field. The number of crossings that the signal makes of a given signal level and the average duration of fades below a given signal level have been derived theoretically for these two cases using a simple statistical model. Comparing the number of level crossings of the electric field with that of the energy density, it is shown that the energy density fades less frequently than the electric field by at least a factor of two. The average duration of fades of the electric field is greater than that of the energy density only for lower signal levels. These results are in reasonable agreement with experimental measurements.*

## I. INTRODUCTION

The study of signal fading appears to be very important to mobile radio systems. When a steady sine wave is sent out from a fixed station, the signal received by a mobile receiver in motion fluctuates, or, in radio jargon, fades. The received signal fluctuates more rapidly as both the frequency of the transmitted wave and the speed of the mobile radio increase. For a field received by a moving isotropic antenna, the maximum fading frequency $f_d$, as Ossanna[1] has pointed out, is $f_d = 2V/\lambda$, where $V$ is the speed of the mobile radio and $\lambda$ is the wavelength of the steady sine wave. For instance, at 836 MHz and a speed of 15 miles/hr, the signal fades at a rate of about 40 times every second and is a serious disturbance to the mobile radio communication.

There have been many investigations of the fading problem. Aikens and Lacy[2] made a test using 450-MHz transmission to a mobile receiver

in an urban area. Bullington[3] investigated radio propagation variation at VHF and UHF. Young[4] pointed out that for the test samples of signal strength taken over a small area, the amplitude follows a Rayleigh distribution to a fair approximation. Also S. O. Rice[5] pointed out that the fluctuations of a received radio signal have the same behavior as the envelope of a narrow-band Gaussian noise. Recently, Ossanna[1] measured the power spectra of a mobile radio fading signal. They all treated the signal as obtained from an isotropic antenna.

In this paper, a theoretical analysis of fading using an energy density antenna[6] is developed and compared with that from an isotropic antenna. The concept of using the energy density antenna to reduce the effect of signal fading was suggested by J. R. Pierce.[7] It will be discussed in detail later. The number of crossings $n(\Psi)$ that the signal makes of a given signal level $\Psi$, and the average duration of fades $\mathrm{t}(\Psi)$ below a given signal level $\Psi$, have been derived theoretically from a statistical model using Gaussian random amplitudes and equal angles of arrival of an infinite number of incoming waves. The two statistical factors, $n$ and t, first expressed by Rice,[8] can describe the property of individual signal fading very well. In this paper, $n$ and t for the isotropic antenna will be compared with the values for the energy density antenna. These theoretical results also will be compared with the experimental data.

## II. THE METHOD OF OBTAINING THE EXPECTED NUMBER OF LEVEL CROSSINGS AND AVERAGE DURATION OF FADES

From Kac's[9] and Rice's[8] paper, a brief derivation of the expected number of level crossings $n(\Psi)$ of a given signal level $\Psi$ and average duration of fades below a given signal level $\Psi$ is as follows. We assume a random function $\psi$ which is statistically stationary in time, and for which the joint probability density function of $\psi$ and its slope $\dot{\psi}$ is $p(\psi, \dot{\psi})$. Any given slope $\dot{\psi}$ can be obtained by

$$\dot{\psi} = \frac{d\psi}{\tau}, \tag{1}$$

where $\tau$ is the time required for a change of ordinate $d\psi$, as shown in Fig. 1. The expected number of crossings of a random function $\psi$ in the interval $(\Psi, \Psi - d\psi)$ for a given slope $\dot{\psi}$ in time $dt$ is

$$\left.\frac{\text{the expected amount of time spent in the interval } d\psi \text{ for a given } \dot{\psi} \text{ in time } dt}{\text{the time required to cross once for a given } \dot{\psi} \text{ in the interval } d\psi}\right|_{\psi=\Psi}$$

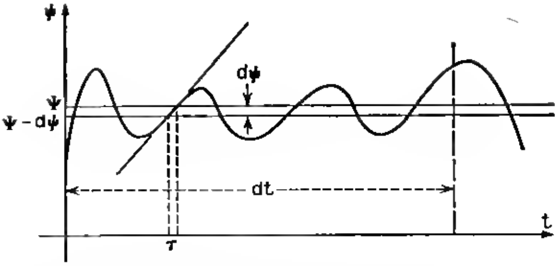


Fig. 1 — The notation used in obtaining the expected number of level crossings $n(\Psi)$ and the average duration of fades $t(\Psi)$.

$$= \frac{E(t)}{\tau} = \frac{p(\psi,\ \dot{\psi})d\psi d\dot{\psi}dt}{\dfrac{d\psi}{\dot{\psi}}}\Bigg|_{\text{at } \psi=\Psi} = \dot{\psi}p(\Psi,\ \dot{\psi})d\dot{\psi}dt. \tag{2}$$

The expected number of crossings for a given $\dot{\psi}$ in time $T$ is

$$\int_0^T \dot{\psi}p(\Psi,\ \dot{\psi})d\dot{\psi}dt = \dot{\psi}p(\Psi,\ \dot{\psi})d\dot{\psi}T. \tag{3}$$

The total expected number of upward crossings in time $T$ is

$$N(\Psi) = T\int_0^\infty \dot{\psi}p(\Psi,\ \dot{\psi})d\dot{\psi}. \tag{4}$$

The total expected number of crossings per second is

$$n(\Psi) = \frac{N(\Psi)}{T} = \int_0^\infty \dot{\psi}p(\Psi,\ \dot{\psi})d\dot{\psi}. \tag{5}$$

Since the expected number of crossings at a particular level $\Psi$ per second can also be stated as

$$n(\Psi) = \frac{\text{the expected amount of time where the function } \psi \text{ is below level } \Psi \text{ in one second}}{\text{the average duration of fades below level } \Psi}$$

$$= \frac{P(\psi < \Psi)}{t(\Psi)}, \tag{6}$$

hence, the average duration of fades below level $\Psi$ is

$$t(\Psi) = \frac{P(\psi < \Psi)}{n(\Psi)}. \tag{7}$$

Hence, the results will he derived from the joint prohahility density function $p(\psi, \dot{\psi})$, and the problem is to derive this prohahility density function for the various signals.

## III. THE EXPECTED NUMBER OF LEVEL CROSSINGS AND THE AVERAGE DURATION OF FADES FOR A VERTICALLY POLARIZED WAVE

In order to obtain the expected numher of level crossings of a given signal level $R$ and the average duration of fades below a given signal level $R$ for the three field components of a vertically polarized wave, first we need to specify the forms of the three field components. Then a statistical model of the field components is assumed. From such a model, we find the joint probability density functions of amplitude $R$ and its slope $\dot{R}$ for the three field components. Finally, we use (5) and (7) to obtain the result for each field component.

Following Gilbert[10] a vertically polarized plane wave $E_z$ traveling in a direction $\mathbf{u}$ in the $(x, y)$ plane is assumed. The three field components referenced to a receiver moving with velocity vector $\mathbf{V}$ can be written

$$E_z = e_z = A_u \exp(-j\beta\mathbf{u}\cdot\mathbf{V}t) \exp(j\omega t) \text{ volt}/m$$

$$H_x = \eta(h_x \text{ amp}/m) = A_u \sin\theta_u \exp(-j\beta\mathbf{u}\cdot\mathbf{V}t) \exp(j\omega t) \text{ volt}/m$$

$$H_y = \eta(h_y \text{ amp}/m) = -A_u \cos\theta_u \exp(-j\beta\mathbf{u}\cdot\mathbf{V}t) \exp(j\omega t) \text{ volt}/m,$$

where $\beta$ is a wave number and $A_u$ is a complex amplitude of an electric wave propagating at a direction $\mathbf{u}$. $\mathbf{u}$ is a unit vector related to an angle $\theta_u$ hetween the positive $x$-axis and the unit vector itself. $\eta$ is free-space wave impedance. The time variation $\exp j\omega t$ can he dropped out of three field components for simplifying the derivation. Moreover, from now on, we will treat the units of all three components $E_z$, $H_x$, and $H_y$ in volt/$m$ which will also simplify the calculation.

When $N$ vertical polarized waves coming from $N$ directions are received by an isotropic antenna of the mobile radio, the three components hecome

$$E_z = \sum_{u=1}^{N} A_u \exp(-j\beta\mathbf{u}\cdot\mathbf{V}t) = \sum_{u=1}^{N} A_u \exp[-j\beta Vt\cos(\theta_u - \alpha)] \quad (8)$$

$$H_x = \sum_{u=1}^{N} A_u \sin\theta_u \exp(-j\beta\mathbf{u}\cdot\mathbf{V}t)$$

$$= \sum_{u=1}^{N} A_u \sin\theta_u \exp[-j\beta Vt\cos(\theta_u - \alpha)] \quad (9)$$

$$H_y = \sum_{u=1}^{N} - A_u \cos\theta_u \exp(-j\beta\mathbf{u}\cdot\mathbf{V}t)$$

$$= -\sum_{u=1}^{N} A_u \cos\theta_u \exp[-j\beta Vt \cos(\theta_u - \alpha)], \tag{10}$$

where $\theta_u$ is the angle between the positive $x$-axis and the direction of $u$th wave **u**, and $0 \leqq \theta_u \leqq 2\pi$. $\alpha$ is the angle between the $x$-axis and the velocity **V**, and $0 \leqq \alpha \leqq 2\pi$. Both $\theta_u$ and $\alpha$ are shown in Fig. 2.

In this paper, a statistical model is used as follows: The complex amplitude $A_u$ can be separated into a real and an imaginary part $A_u = R_u + jS_u$, hence $N$ incoming waves have $N$ real values of $R_u$ and $S_u$. We suppose all those $2N$ real values are Gaussian independent variables with mean zero and variance one. Also, we assume the $N$ waves have uniform angular distribution, i.e., the $k$th wave $u_k$ has an angle of arrival $\theta_u = 2\pi k/N$. Moreover, in this paper an infinite number of multiply reflected waves ($N \rightarrow \infty$) are assumed for finding the expected number of level crossings $n(R)$ of a given signal level $R$, and the average duration $\mathbf{t}(R)$ of fades below a given signal amplitude $R$.

### 3.1 *Finding the Values of $n(R)$ and $\mathbf{t}(R)$ from the $E_z$ Field*

First of all, we need to obtain the joint probability density function of signal amplitude $R$ and its slope $\dot{R}$ for the electric field component $E_z$ using the statistical model we mentioned previously. We start from (8). The alternate form of (8) can be written as

$$E_z = \sum_{u=1}^{N} (R_u + jS_u)[\cos\{\beta Vt \cos(\theta_u - \alpha)\} - j\sin\{\beta Vt\cos(\theta_u - \alpha)\}]. \tag{11}$$

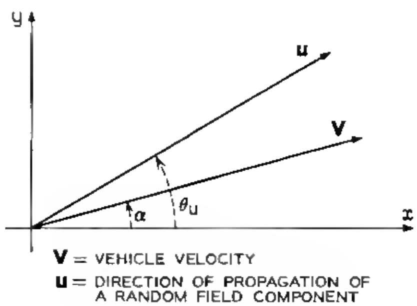


Fig. 2 — The coordinate system.

Equation (11) can be separated into real and imaginary parts

$$E_z = X_1 + jY_1 . \tag{12}$$

The real part of $E_z$ is

$$X_1 = \sum_{\mu=1}^{N} (R_u \cos \varphi_u + S_u \sin \varphi_u) \tag{13}$$

and the imaginary part of $E_z$ is

$$Y_1 = \sum_{u=1}^{N} (S_u \cos \varphi_u - R_u \sin \varphi_u), \tag{14}$$

where

$$\varphi_u = \beta V t \cos (\theta_u - \alpha). \tag{15}$$

We assume that all $N$ values of $R$ and $S$ in (13) and (14) are time independent. Then the derivatives with respect to time of (13) and (14) are

$$\dot{X}_1 = \beta V \sum_{u=1}^{N} (-R_u \sin \varphi_u + S_u \cos \varphi_u) \cos (\theta_u - \alpha) \tag{16}$$

$$\dot{Y}_1 = \beta V \sum_{u=1}^{N} (-S_u \sin \varphi_u - R_u \cos \varphi_u) \cos (\theta_u - \alpha). \tag{17}$$

The mean values, variances, and covariances of $X_1$, $Y_1$, $\dot{X}_1$, and $\dot{Y}_1$ are

$$m_1 = \langle X_1 \rangle = \langle Y_1 \rangle = \langle \dot{X}_1 \rangle = \langle \dot{Y}_1 \rangle = 0$$

$$\mu_{11} = \langle X_1^2 \rangle = \langle Y_1^2 \rangle = N \quad \text{for any} \quad N \tag{18}$$

$$\mu_{11}' = \langle \dot{X}_1^2 \rangle = \langle \dot{Y}_1^2 \rangle = (\beta V)^2 \frac{N}{2} \quad \text{for} \quad N \geqq 3 \tag{19}$$

and

$$\langle X_1 Y_1 \rangle = \langle X_1 \dot{X}_1 \rangle = \langle Y_1 \dot{X}_1 \rangle = \langle Y_1 \dot{Y}_1 \rangle = \langle \dot{X}_1 \dot{Y}_1 \rangle = \langle X_1 \dot{Y}_1 \rangle = 0.$$

The above results are shown in the Appendix.

From the central limit theorem, it follows that $X_1$, $Y_1$, $\dot{X}_1$, and $\dot{Y}_1$ are four independent random variables which are distributed normally as the value $N$ approaches infinity. The probability density function of four independent real random variables $X_1$, $Y_1$, $\dot{X}_1$, and $\dot{Y}_1$ is[11]

$$p(X_1, Y_1, \dot{X}_1, \dot{Y}_1) = \frac{1}{(2\pi)^2 |\mu|^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} \left( \frac{X_1^2 + Y_1^2}{\mu_{11}} + \frac{\dot{X}_1^2 + \dot{Y}_1^2}{\mu_{11}'} \right) \right\}, \tag{20}$$

where the determinant $|\mu|$ of the covariance matrix is

$$|\mu| = (\mu_{11}\mu_{11}')^2 = \frac{N^4}{4}(\beta V)^4.$$

We may introduce the concept of the envelope

$$E_s = X_1 + jY_1 = r_e \underline{/\eta_e} .$$

The quantity $r_e$ is the envelope and $\eta_e$ is the phase, both of which are slowly varying functions of the time. Then,

$$X_1 = r_e \cos \eta_e \quad ; \quad Y_1 = r_e \sin \eta_e \tag{21a}$$

$$\dot{X}_1 = \dot{r}_e \cos \eta_e - r_e \dot{\eta}_e \sin \eta_e \; ; \quad \dot{Y}_1 = \dot{r}_e \sin \eta_e + r_e \dot{\eta}_e \cos \eta_e \, . \tag{21b}$$

The Jacobian of the transformation from $(X_1 , Y_1 , \dot{X}_1 , \dot{Y}_1)$-space to $(r_e , \eta_e , \dot{r}_e , \dot{\eta}_e)$-space is[12] $| J | = r_e^2$.

Therefore, the change of variables gives the probability density the form

$$\begin{aligned} p(X_1 , Y_1 , \dot{X}_1 , \dot{Y}_1) &= r_e^2 q(r_e , \eta_e , \dot{r}_e , \dot{\eta}_e) \\ &= p(r_e , \eta_e , \dot{r}_e , \dot{\eta}_e) \\ &= \frac{r_e^2}{(2\pi)^2 | \mu |^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\left(\frac{r_e^2}{\mu_{11}} + \frac{r_e^2 \dot{\eta}_e^2 + \dot{r}_e^2}{\mu_{11}'}\right)\right\}, \end{aligned} \tag{22}$$

where $q(r_e , \eta_e , \dot{r}_e , \dot{\eta}_e)$ is the density obtained on substituting for $X_1$ , $Y_1$ , etc., their values in $r_e$ , $\eta_e$ , etc., obtained from (21a) and its time derivative (21b). To obtain $p(r_e , \dot{r}_e)$, the probability density of the envelope and its rate of change, we must integrate over $\eta_e$ and $\dot{\eta}_e$ , the range of which are, respectively, $(0 \text{ to } 2\pi)$ and $(-\infty \text{ to } +\infty)$. From (22) we obtain

$$p(r_e , \dot{r}_e) = \frac{r_e}{\sqrt{2\pi\mu_{11}'\mu_{11}}} \exp\left\{\frac{1}{2}\left(\frac{r_e^2}{\mu_{11}} + \frac{\dot{r}_e^2}{\mu_{11}'}\right)\right\}. \tag{23}$$

It is observed that the expression on the right of (23) is independent of $t$. Hence, the expected number of level crossings $n(R_e)$ at a given signal amplitude $(r_e = R_e)$ can be obtained from (5) by using $p(R_e , \dot{r}_e)$ in (23).

$$n(R_e) = \int_0^\infty \dot{r}_e p(R_e , \dot{r}_e) d\dot{r}_e = \sqrt{\frac{\mu_{11}'}{\mu_{11}}} \frac{R_e}{\sqrt{2\pi\mu_{11}}} \exp\left(-\frac{R_e^2}{2\mu_{11}}\right). \tag{24}$$

Now the variance of $r_e$ is

$$\langle r_e^2 \rangle = \langle X_1^2 \rangle + \langle Y_1^2 \rangle = 2\mu_{11} = 2N.$$

Let

$$\tilde{R}_e = R_e/\sqrt{\langle r_e^2 \rangle} = R_e/r_{e(rms)} = R_e/\sqrt{2N}. \tag{25}$$

Substituting the values of variances $\mu_{11}$ and $\mu_{11}'$ from (18) and (19) into (24), also applying the relations in (25), we obtain

$$n(\tilde{R}_e) = \frac{\beta V}{\sqrt{2\pi}} \tilde{R}_e \exp(-\tilde{R}_e^2). \tag{26}$$

Equation (26) is plotted in Fig. 3 where the abscissa is $\tilde{R}_e$ in dB (20 log $\tilde{R}_e$) and the ordinate is $(\sqrt{2\pi}/\beta V)n(\tilde{R}_e)$.

The average duration of fades $\mathbf{t}(R_e)$ of $E_z$ can be obtained as follows: The probability that the envelope $r_e$ is less than a given amplitude level $R_e$ is

$$P(r_e(t) < R_e) = \int_0^{R_e} p(r_e(t))dr_e(t), \tag{27}$$

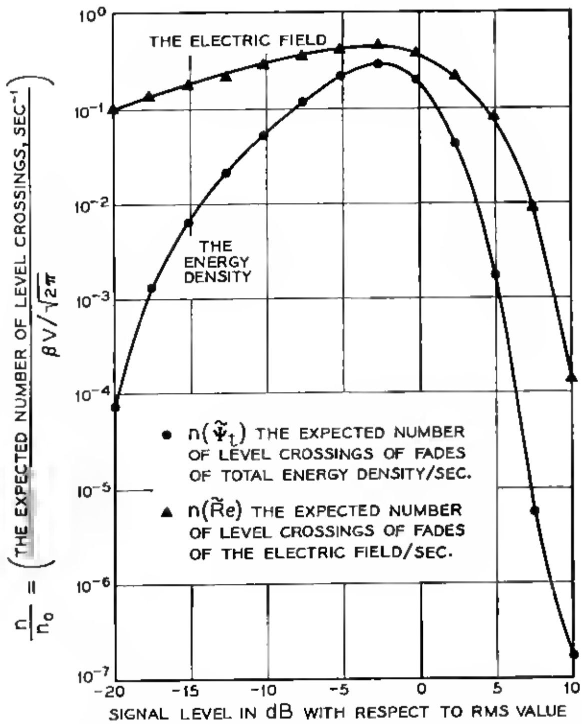


Fig. 3 — Comparison of the level crossing rate of the electric field with that of the energy density.

where $p(r_e(t))$ is the probability density obtained from $p(X_1, Y_1)$. By changing the variables, we obtain $p(r_e(t), \eta_e(t))$ from $p(X_1, Y_1)$. Then we integrate $p(r_e(t), \eta_e(t))$ over from 0 to $2\pi$ to obtain[13]

$$p(r_e(t)) = \frac{r_e(t)}{\mu_{11}} \exp\left[-\frac{r_e^2(t)}{2\mu_{11}}\right], \tag{28}$$

where $\mu_{11}$ is obtained from (18). Substituting (28) into (27) we get

$$P(r_e(t) < R_e) = 1 - \exp(-R_e^2/2\mu_{11}) = 1 - \exp(-\tilde{R}_e^2). \tag{29}$$

The expected number of times per second that $r_e(t)$ passes upward (or downward) across the level $R_e$ is obtained from (24). The average duration of fades during which $r_e(t) < R_e$ may then be obtained by substituting (24) and (29) into (7)

$$\mathfrak{t}(R_e) = \frac{P(r_e(t) < R_e)}{n(R_e)} = \frac{P(\tilde{r}_e(t) < \tilde{R}_e)}{n(\tilde{R}_e)} = \frac{\sqrt{2\pi}}{\beta V}\frac{1}{\tilde{R}_e}[\exp(\tilde{R}_e^2) - 1] \tag{30}$$

which is shown in Fig. 4.

### 3.2 *Finding the Values of $n(R_{hz})$ and $\mathfrak{t}(R_{hz})$ from the $H_z$ Field*

Following the same steps as above, we are going to find the joint probability density function $p(r_{hz}, \dot{r}_{hz})$ of the envelope $r_{hz}$ and its slope $\dot{r}_{hz}$ of the $H_z$ field component first. From (9) we find the real and imaginary parts of $H_z$ which are expressed in the Appendix. The means, variances, and covariances of four real Gaussian random variables shown in the Appendix are

$$m_2 = \langle X_2 \rangle = \langle Y_2 \rangle = \langle \dot{X}_2 \rangle = \langle \dot{Y}_2 \rangle = 0$$

$$\mu_{22} = X_2^2 = Y_2^2 = \frac{N}{2} \qquad \text{for any } N$$

$$\mu_{22}' = \dot{X}_2^2 = \dot{Y}_2^2 = \frac{N}{8}(\beta V)^2$$

$$\cdot[\cos^2\alpha + 3\sin^2\alpha] \quad \text{for} \quad N = 3 \quad \text{and} \quad N \geqq 5$$

$$\langle X_2 Y_2 \rangle = \langle X_2 \dot{X}_2 \rangle = \langle X_2 \dot{Y}_2 \rangle = \langle Y_2 \dot{X}_2 \rangle = \langle Y_2 \dot{Y}_2 \rangle = \langle \dot{X}_2 \dot{Y}_2 \rangle = 0.$$

The probability density of the envelope of $H_z$ field and its rate of change $p(r_{hz}, \dot{r}_{hz})$ is then obtained by following the same procedure used in deriving $p(r_e, \dot{r}_e)$.

$$p(r_{hz}, \dot{r}_{hz}) = \frac{1}{\sqrt{2\pi\mu_{22}'}}\frac{r_{hz}}{\mu_{22}} \exp\left\{-\frac{1}{2}\left(\frac{r_{hz}^2}{\mu_{22}} + \frac{\dot{r}_{hz}^2}{\mu_{22}'}\right)\right\}. \tag{31}$$

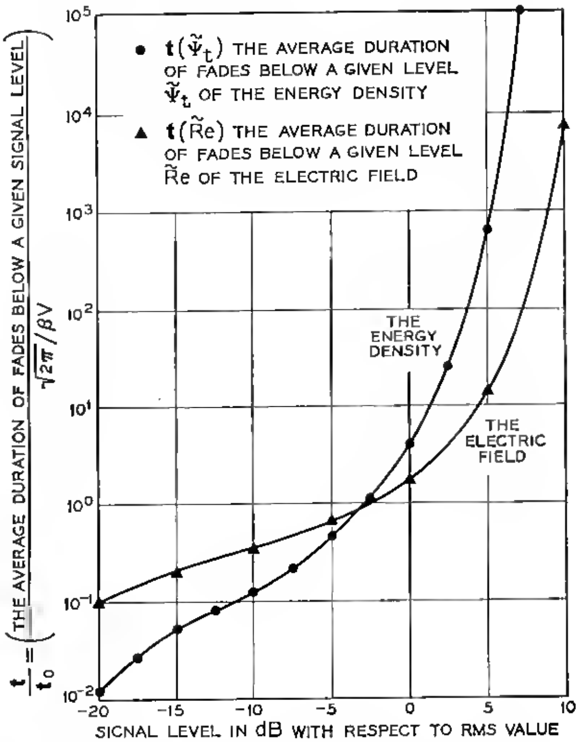


Fig. 4 — Comparison of the duration of fades of the electric field with that of the energy density.

Hence, the expected number of level crossings $n(R_{hz})$ at a given signal amplitude ($r_{hz} = R_{hz}$) can be obtained from (5) by using $p(R_{hz}, \dot{r}_{hz})$ in (31)

$$n(R_{hz}) = \int_0^\infty \dot{r}_{hz} p(R_{hz}, \dot{r}_{hz}) d\dot{r}_{hz} = \sqrt{\frac{\mu_{22}'}{\mu_{22}}} \left(\frac{R_{hz}}{\sqrt{2\pi\mu_{22}}}\right) \exp\left(-\frac{R_{hz}^2}{2\mu_{22}}\right). \quad (32)$$

The variance of $r_{hz}$ is

$$\langle r_{hz}^2 \rangle = \langle X_2^2 \rangle + \langle Y_2^2 \rangle = 2\mu_{22} = N.$$

Let

$$\tilde{R}_{hz} = \frac{R_{hz}}{\sqrt{\langle r_{hz}^2 \rangle}} = \frac{R_{hz}}{r_{hz(rms)}} = \frac{R_{hz}}{\sqrt{N}}. \quad (33)$$

Substituting the values of the variances $\mu_{22}$ and $\mu'_{22}$ into (32) and replacing $N$ by $\langle r_{hz}^2 \rangle$ we get

$$n(\tilde{R}_{hz}) = \frac{\beta V}{\sqrt{2\pi}} \sqrt{\frac{\cos^2 \alpha + 3 \sin^2 \alpha}{2}}\, \tilde{R}_{hz} \exp -\tilde{R}_{hz}^2$$

$$= \frac{\beta V}{\sqrt{2\pi}} \sqrt{1 - \tfrac{1}{2} \cos 2\alpha}\, \tilde{R}_{hz} \exp -\tilde{R}_{hz}^2 . \tag{34}$$

Equation (34) is the same form as (26) except for a multiplying factor which is a function of $\alpha$ shown in Fig. 5. Hence, $n(\tilde{R}_{hz})$ is also a function of angle $\alpha$. Thus, when the mobile is moving along the $x$-axis $\alpha = 0$ or $\pi$, and

$$n(\tilde{R}_{hz}) = \frac{\beta V}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} \tilde{R}_{hz} \exp -\tilde{R}_{hz}^2 \tag{34a}$$

which is the minimum value of $n(\tilde{R}_{hz})$. When the mobile is moving on $\pm y$-axis $\alpha = \pm\pi/2$, and

$$n(\tilde{R}_{hz}) = \frac{\beta V}{\sqrt{2\pi}} \sqrt{\frac{3}{2}}\, \tilde{R}_{hz} \exp -\tilde{R}_{hz}^2 \tag{34b}$$

which is the maximum value of $n(\tilde{R}_{hz})$. The ratio of level crossings for these two cases is

$$\frac{n(\tilde{R}_{hz})(\alpha = 0°,\ 180°)}{n(\tilde{R}_{hz})(\alpha = \pm 90°)} = \frac{1}{\sqrt{3}}. \tag{34c}$$

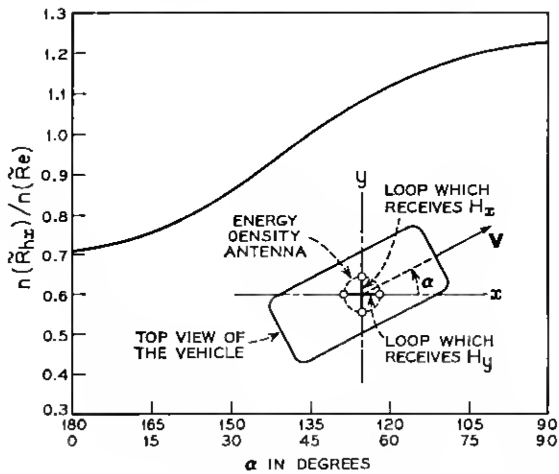


Fig. 5 — The effect of the angle $\alpha$ on the ratio of level crossing rates of the electric field to the $x$-component of the magnetic field.

For $\alpha = \pm 45°$ and $\pm 135°$,

$$n(\tilde{R}_{hx}) = \frac{\beta V}{\sqrt{2\pi}} \tilde{R}_{hx} \exp -\tilde{R}_{hx}^2 \tag{34d}$$

which is the same expression as (26) for $\tilde{R}_e$. However, although (34d) and (26) are the same form, the magnitudes $\tilde{R}_e$ and $\tilde{R}_{hx}$ are different, since $\langle r_e^2 \rangle$ in (25) is equal to two times $\langle r_{hx}^2 \rangle$ in (33).

The average duration of fades $\mathsf{t}(R_{hx})$ of $H_x$ can be obtained without difficulty. It is easy to prove that the expression for the average length of the intervals during which $r_{hx} < R_{hx}$ will have the same form as $\mathsf{t}(R_e)$ in (30), except for a multiplying factor that depends on $\alpha$, as follows:

$$\mathsf{t}(\tilde{R}_{hx}) = \frac{\sqrt{2\pi}}{\beta V} \frac{1}{\sqrt{1 - \frac{1}{2}\cos 2\alpha}} \frac{1}{\tilde{R}_{hx}} [\exp(\tilde{R}_{hx}^2) - 1]. \tag{35}$$

When $\alpha = 0°$ and $180°$:

$$\mathsf{t}(\tilde{R}_{hx}) = \frac{\sqrt{2\pi}}{\beta V} \sqrt{2} \frac{1}{\tilde{R}_{hx}} [\exp(\tilde{R}_{hx}^2) - 1] \tag{35a}$$

which is the maximum value of $\mathsf{t}(R_{hx})$, and when $\alpha = \pm 90°$:

$$\mathsf{t}(\tilde{R}_{hx}) = \frac{\sqrt{2\pi}}{\beta V} \sqrt{\frac{2}{3}} \frac{1}{\tilde{R}_{hx}} [\exp(\tilde{R}_{hx}^2) - 1] \tag{35b}$$

which is the minimum value of $\mathsf{t}(R_{hx})$. The ratio of level crossings for these two cases is

$$\frac{\mathsf{t}(\tilde{R}_{hx})(\alpha = 0°, 180°)}{\mathsf{t}(\tilde{R}_{hx})(\alpha = \pm 90°)} = \sqrt{3} \tag{35c}$$

which is the inverse of (34c). This tells us that when $n(\tilde{R}_{hx})$ reaches a maximum value, the average duration of fades reaches a minimum value and vice versa.

For $\alpha = \pm 45°$, $\pm 135°$

$$\mathsf{t}(\tilde{R}_{hx}) = \frac{2\pi}{\beta V} \frac{1}{\tilde{R}_{hx}} [\exp(\tilde{R}_{hx}^2) - 1] \tag{35d}$$

which is the same form as the expression for $\mathsf{t}(\tilde{R}_e)$ in (30). We may say at these angles $\alpha = \pm 45°$ and $\pm 135°$, the $E$ field and the $H_x$ field bave the same average duration of fades below the level $\tilde{R}_e = \tilde{R}_{hx}$.

### 3.3 *Finding the Values of* $n(R_{hy})$ *and* $\mathsf{t}(R_{hy})$ *from the* $H_y$ *Field*

Similarly, we obtain $n(R_{hy})$, the expected number of level crossings of a given signal level $R_{hy}$ from the $H_y$ field. It is very easy to see that $n(R_{hx})$ and $n(R_{hy})$ are the same forms of distribution as expressed in (34), except for the multiplying factor that depends on $\alpha$. The average duration of fades $\mathsf{t}(R_{hy})$ during which $r_{hy}(t) < R_{hy}$ is also of the same form as $\mathsf{t}(R_{hx})$ in (35) except for the multiplying factor depending on $\alpha$. The variances $\mu_{33}$ and $\mu_{33}'$ are given in the Appendix. We may thus write directly

$$n(\tilde{R}_{hy}) = \frac{\beta V}{\sqrt{2\pi}} \sqrt{1 + \tfrac{1}{2} \cos 2\alpha}\, \tilde{R}_{hy} \exp -\tilde{R}_{hy}^2 \tag{36}$$

$$\mathsf{t}(\tilde{R}_{hy}) = \frac{\sqrt{2\pi}}{\beta V} \frac{1}{\sqrt{1 + \tfrac{1}{2} \cos 2\alpha}} \frac{1}{\tilde{R}_{hy}} [\exp(\tilde{R}_{hy}^2) - 1], \tag{37}$$

where

$$\tilde{R}_{hy} = \frac{R_{hy}}{\sqrt{\langle r_{hy}^2 \rangle}} = \frac{R_{hy}}{r_{hy(rms)}}.$$

It is obvious that

$$n(\tilde{R}_{hy})_{\alpha=0^\circ,180^\circ} = n(\tilde{R}_{hx})_{\alpha=\pm 90^\circ}$$

$$n(\tilde{R}_{hy})_{\alpha=\pm 90^\circ} = n(\tilde{R}_{hx})_{\alpha=0^\circ,180^\circ}$$

when $\tilde{R}_{hy} = \tilde{R}_{hx}$.

## IV. THE EXPECTED NUMBER OF LEVEL CROSSINGS AND THE AVERAGE DURATION OF FADES OF THE SIGNAL FROM AN ENERGY DENSITY ANTENNA

J. R. Pierce[7] has suggested utilization of the energy density concept as a possible means for reducing the signal fading in mobile radio. If we pick up the electric field $e$ and the magnetic field $h$ in free space and amplify the two fields by their appropriate relative gains, square and add these two fields, we obtain a signal proportional to electromagnetic energy density

$$W = \tfrac{1}{2}(\epsilon e^2 + \mu h^2), \tag{38}$$

where $\epsilon$ is dielectric constant, and $\mu$ is permeability. This idea can be realized by using a special antenna[6] which receives three field components $e_z$, $h_x$, and $h_y$ simultaneously. The three signals enter separate

square-law detectors, and the three detector outputs are added to obtain the energy density,

$$W = \tfrac{1}{2}(\epsilon \, |e_z|^2 + \mu \, |h_x|^2 + \mu \, |h_y|^2).$$

We may express $W$ in a different form

$$\begin{aligned} W &= \frac{\epsilon}{2}\left[\left(|e_z|^2 + \frac{\mu}{\epsilon}\,|h_x|^2 + \frac{\mu}{\epsilon}\,|h_y|^2\right) \text{volt}^2/m^2\right] \\ &= \frac{\epsilon}{2}\,[|E_z|^2\,(\text{volt}^2/m^2) + |H_x|^2\,(\text{volt}^2/m^2) + |H_y|^2\,(\text{volt}^2/m^2)] \\ &= \frac{\epsilon}{2}\,[\psi_t(\text{volt}^2/m^2)] = \frac{\epsilon\psi_t}{2}\ \text{Joules}/m^3. \end{aligned} \tag{39}$$

We define $\psi_t$ as a normalized energy density

$$\psi_t = |E_z|^2 + |H_x|^2 + |H_y|^2\, \text{volt}^2/m^2$$

$$= \psi_e + \psi_{hx} + \psi_{hy} \tag{40}$$

$$= (X_1^2 + Y_1^2) + (X_2^2 + Y_2^2) + (X_3^2 + Y_3^2). \tag{41}$$

Gilbert[10] has done some work on finding power spectra in energy reception for mobile radio. His work provides very useful background for this paper.

In this section, we are attempting to derive the number of crossings $n(\Psi_t)$ at a given level of signal magnitude $\Psi_t$ using (5) in Section II. First of all, we need to find the joint probability density function $p(\psi_t, \dot{\psi}_t)$ of signal $\psi_t$ and its slope $\dot{\psi}_t$. Since $\psi_t$ is a function of $(X_1, Y_1, X_2, Y_2, X_3, Y_3)$, and $\dot{\psi}_t$ is assumed to be a function of $(\dot{\psi}_e, \dot{\psi}_{hx}, \dot{\psi}_{hy})$, we will find out that the variables $(X_1, Y_1, X_2, Y_2, X_3, Y_3)$ and $(\dot{\psi}_e, \dot{\psi}_{hx}, \dot{\psi}_{hy})$ are two independent Gaussian variable groups. Then,

$$\begin{aligned} &p[\psi_t(X_1, Y_1, X_2, Y_2, X_3, Y_3), \dot{\psi}_t(\dot{\psi}_e, \dot{\psi}_{hx}, \dot{\psi}_{hy})] \\ &\quad = p[\psi_t(X_1, Y_1, X_2, Y_2, X_3, Y_3)] \times p[\dot{\psi}_t(\dot{\psi}_e, \dot{\psi}_{hx}, \dot{\psi}_{hy})] \\ &\quad = p(\psi_t)p(\dot{\psi}_t). \end{aligned} \tag{42}$$

A brief sketch of the method of finding $p(\psi_t)$ and $p(\dot{\psi}_t)$ is discussed below.

### 4.1 *To get $p(\psi_t)$*

Since we know from (41) that

$$\psi_t = (X_1^2 + Y_1^2) + (X_2^2 + Y_2^2) + (X_3^2 + Y_3^2)$$

and since $X_1$, $Y_1$, $\cdots$ are independent Gaussian variables, it is easy to get $p(\psi_t)$ from the Fourier transform of the characteristic function $M_{\psi_t}(v)$, where $M_{\psi_t}(v) = M_{X_1^2} M_{Y_1^2} M_{X_2^2} M_{Y_2^2} \cdots$ etc., and we can get these $M$'s very easily.

### 4.2 *To get $p(\dot{\psi}_t)$*

All the terms in the summations of equations (79), (80), and (81) which represent $\dot{\psi}_e$, $\dot{\psi}_{hx}$ and $\dot{\psi}_{hy}$, respectively, in the Appendix are statistically independent. Then by the central limit theorem these three variables $\dot{\psi}_e$, $\dot{\psi}_{hx}$, and $\dot{\psi}_{hy}$ are Gaussian distributed. Hence, the joint probability density function $p(\dot{\psi}_e, \dot{\psi}_{hx}, \dot{\psi}_{hy})$ can be established. Since $\dot{\psi}_t = \dot{\psi}_e + \dot{\psi}_{hx} + \dot{\psi}_{hy}$ we can get $p(\dot{\psi}_t)$ from the Fourier transform of the characteristic function $M_{\dot{\psi}_t}$, but $M_{\dot{\psi}_t}$ must now be obtained from the general definition

$$M_{\dot{\psi}_t}(v) = E[e^{jv\dot{\psi}_t}] = \iiint e^{jv\dot{\psi}_t} p(\dot{\psi}_e, \dot{\psi}_{hx}, \dot{\psi}_{hy}) d\dot{\psi}_e d\dot{\psi}_{hx} d\dot{\psi}_{hy} \tag{43}$$

since we have no simple way of getting $M_{\dot{\psi}_e}$, $M_{\dot{\psi}_{hx}}$ and $M_{\dot{\psi}_{hy}}$ separately.

Let us introduce a new variable $\epsilon$ which can be any one of the above Gaussian random variables. It has a zero mean and variance $\mu$. Then the probability density function of the square $\epsilon^2$ is[14]

$$p(\gamma = \epsilon^2) = \frac{1}{\sqrt{2\pi\mu\gamma}} \exp\left(-\frac{1}{2\mu}\gamma\right) \tag{44}$$

for $\gamma > 0$. The characteristic function corresponding to this probability density is

$$M_\gamma(jv) = \int_0^\infty e^{jv\gamma} p(\gamma) d\gamma = (1 - j2\mu v)^{-\frac{1}{2}}. \tag{45}$$

From the Appendix we know all six variables $X_1$, $Y_1$, $X_2$, $Y_2$, $X_3$ and $Y_3$ are independent Gaussian variables. It is not hard to see that the $X_1^2$, $Y_1^2$, $X_2^2$, $Y_2^2$, $X_3^2$, $Y_3^2$ are independent variables by obtaining $p(X_1^2, Y_1^2, X_2^2, Y_2^2, X_3^2, Y_3^2)$ from the Jacobian of the transformation[12] of $p(X_1, Y_1, X_2, Y_2, X_3, Y_3)$. Then $X_1^2$ and $Y_1^2$ have the same characteristic function $(1 - 2j\mu_{11}v)^{-\frac{1}{2}}$. Also $X_2^2$, $Y_2^2$, $X_3^2$, and $Y_3^2$ have the same characteristic function $(1 - 2j\mu_{22}v)^{-\frac{1}{2}}$. Thus, by the addition theorem[15] the sum $\psi_t$ which is defined in (41) has the characteristic function

$$M_{\psi_t}(jv) = \frac{1}{(1 - 2j\mu_{11}v)(1 - 2j\mu_{22}v)^2}. \tag{46}$$

Then the probability density function $p(\psi_t)$ can be obtained from the Fourier transformation of the characteristic function

$$p(\psi_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iv\psi_t} M_{\psi_t}(jv)\,dv$$

$$= \frac{\mu_{11}}{2(\mu_{22}-\mu_{11})^2}\left[\exp\left(-\frac{\psi_t}{2\mu_{11}}\right) - \exp\left(-\frac{\psi_t}{2\mu_{22}}\right)\right] + \frac{\psi_t \exp\left(-\dfrac{\psi_t}{2\mu_{22}}\right)}{4\mu_{22}(\mu_{22}-\mu_{11})}. \tag{47}$$

The joint probability density $p(\dot\psi_e, \dot\psi_{hx}, \dot\psi_{hy})$ has been derived in the Appendix

$$p(\dot\psi_e, \dot\psi_{hx}, \dot\psi_{hy})$$
$$= \frac{1}{(2\pi)^{\frac{3}{2}} |\Lambda|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}(|\Lambda_{11}|\dot\psi_e^2 + |\Lambda_{22}|\dot\psi_{hx}^2 + |\Lambda_{33}|\dot\psi_{hy}$$
$$+ 2|\Lambda_{12}|\dot\psi_e\dot\psi_{hx} + 2|\Lambda_{13}|\dot\psi_e\dot\psi_{hy} + 2|\Lambda_{23}|\dot\psi_{hx}\dot\psi_{hy})\}, \tag{48}$$

where $[\Lambda]$ is the covariance matrix of $\dot\psi_e$, $\dot\psi_{hx}$, $\dot\psi_{hy}$ and the $|\Lambda_{nm}|$ are the cofactors of $|\Lambda|$, given in the Appendix. From (40), $\dot\psi_t$ is the sum of the three random variables $\dot\psi_e$, $\dot\psi_{hx}$, and $\dot\psi_{hy}$. Then the characteristic function for $\dot\psi_t$ is

$$M_{\dot\psi_t}(jv) = E[\exp\{jv(\dot\psi_e + \dot\psi_{hx} + \dot\psi_{hy}\}]$$

$$= \iiint_{-\infty}^{\infty} \exp\{jv(\dot\psi_e + \dot\psi_{hx} + \dot\psi_{hy})\} p(\dot\psi_e, \dot\psi_{hx}, \dot\psi_{hy})\,d\dot\psi_e\,d\dot\psi_{hx}\,d\dot\psi_{hy}. \tag{49}$$

The details of this computation are given in the Appendix with the result (92)

$$M_{\dot\psi_t}(jv) = \exp\{-\tfrac{1}{2}\rho_t' v^2\}, \tag{50}$$

where

$$\rho_t' = \rho_{11}' + \rho_{22}' + \rho_{33}' + 2\rho_{12}' + 2\rho_{13}'. \tag{51}$$

The probability density of total $\dot\psi_t$ is then

$$p(\dot\psi_t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M_{\dot\psi_t}(jv) e^{-iv\dot\psi_t}\,dv = \frac{1}{\sqrt{2\pi\rho_t'}}\exp\left(-\frac{1}{2\rho_t'}\dot\psi_t^2\right). \tag{52}$$

The joint probability density $p(\psi_t, \dot\psi_t)$ of $\psi_t$ and $\dot\psi_t$ can now be obtained by substituting (47) and (52) into (42).

The expected number of crossings $n(\Psi_t)$ at a given signal level $\psi_t = \Psi_t$ made by the total energy density signal in one second can be obtained from (5)

$$n(\Psi_t) = \int_0^\infty \dot{\psi}_t p(\Psi_t, \dot{\psi}_t) d\dot{\psi}_t = p(\Psi_t) \int_0^\infty \dot{\psi}_t p(\dot{\psi}_t) d\dot{\psi}_t$$
$$= \sqrt{\frac{\rho_t'}{2\pi}} \left\{ \frac{\mu_{11}}{2(\mu_{22} - \mu_{11})^2} \left[ \exp\left(-\frac{\Psi_t}{2\mu_{11}}\right) - \exp\left(-\frac{\Psi_t}{2\mu_{22}}\right) \right] + \frac{\Psi_t \exp - \Psi_t/2\mu_{22}}{4\mu_{22}(\mu_{22} - \mu_{11})} \right\}, \qquad (53)$$

where $\rho_t'$ is given by (51) and $\mu_{11} = N$, $\mu_{22} = N/2$ as shown in the Appendix.

Also we know from Gilbert[10]

$$\langle \psi_t^2 \rangle = 22N^2.$$

In addition, $\rho_t' = \rho_{11}' + \rho_{22}' + \rho_{33}' + 2\rho_{12}' + 2\rho_{13}' = 2N^2(\beta V)^2$ and letting

$$\tilde{\Psi}_t = \frac{\Psi_t}{\sqrt{\langle \psi_t^2 \rangle}} = \frac{\Psi_t}{\psi_{t\,(\mathrm{rms})}}$$

we can simplify (53) as follows:

$$n(\tilde{\Psi}_t) = \frac{\beta V}{\sqrt{2\pi}} \left\{ 2\sqrt{2} \left[ \exp\left(-\frac{\sqrt{22}}{2}\tilde{\Psi}_t\right) - \exp\left(-\sqrt{22}\tilde{\Psi}_t\right) \right] - 2\sqrt{11}\tilde{\Psi}_t \exp\left(-\sqrt{22}\tilde{\Psi}_t\right) \right\}. \qquad (54)$$

Equation (54) is a distribution whicb is independent of the angle $\alpha$. When $\tilde{\Psi}_t = 1$, it means that $\Psi_t$ is equal to its rms value. Equation (54) then becomes

$$n(\tilde{\Psi})_{\tilde{\Psi}=1} = \frac{\beta V}{\sqrt{2\pi}} \times 0.1839.$$

Also let $\tilde{R}_e = 1$ in (26):

$$n(\tilde{R}_e)_{\tilde{R}_e=1} = \frac{\beta V}{\sqrt{2\pi}} \times 0.3678.$$

It is shown that tbe expected number of crossings of the total energy density is one half tbe expected number of crossings of the envelope of

$E$ field at their rms levels ($\tilde{\Psi}_t = \tilde{R}_e^2$). In other words, the energy density fades less frequently than the $E$ field by at least a factor of 2.

$$n(\tilde{\Psi}_t) \leqq \tfrac{1}{2} n(\tilde{R}_e)$$

for $\tilde{\Psi}_t = \tilde{R}_e^2$ with respect to their rms values. The theoretical values of $n(\Psi)$ and $n(R_e)$ are in Fig. 3.

The average duration of fades below a given level $\Psi_t$ is given by (7)

$$\mathrm{t}(\Psi_t) = \frac{P(\psi_t(t) < \Psi_t)}{n(\Psi_t)}, \tag{55}$$

where $P(\psi_t(t) < \Psi_t)$ is the probability function obtained from $p(\psi_t(t))$.

$$\begin{aligned} P(\psi_t(t) < \Psi_t) &= \int_0^{\Psi_t} p(\psi_t(t)) d\psi_t(t) \\ &= 1 - 4 \exp\left(-\frac{\sqrt{22}}{2}\tilde{\Psi}_t\right) + (3 + \sqrt{22}\tilde{\Psi}_t) \exp(-\sqrt{22}\tilde{\Psi}_t), \end{aligned} \tag{56}$$

where

$$\tilde{\Psi}_t = \frac{\Psi_t}{\psi_{t\,(\mathrm{rms})}}.$$

Substituting (54) and (56) into (55), we obtain the average duration of a fade below a given level $\Psi_t$:

$$\mathrm{t}(\tilde{\Psi}_t) = \frac{\sqrt{2\pi}}{\beta V} \frac{1}{\sqrt{2}} \cdot \left\{ \frac{1 - 2\exp\left(-\frac{\sqrt{22}}{2}\tilde{\Psi}_t +\right)\exp(-\sqrt{22}\tilde{\Psi}_t)}{2\left[\exp\left(-\frac{\sqrt{22}}{2}\tilde{\Psi}_t\right) - \exp(-\sqrt{22}\tilde{\Psi}_t)\right] - \sqrt{22}\tilde{\Psi}_t \exp(-\sqrt{22}\tilde{\Psi}_t)} - 1 \right\}. \tag{57}$$

When $\tilde{\Psi}_t = 1$, (57) becomes

$$\mathrm{t}(\tilde{\Psi}_t) = \frac{\sqrt{2\pi}}{\beta V} \times 3.74.$$

When $\tilde{R}_e = 1$, (30) becomes

$$\mathrm{t}(\tilde{R}_e) = \frac{\sqrt{2\pi}}{\beta V} \times 1.7183.$$

It is shown that the average duration of fades of the energy density below a level $\tilde{\Psi}_t$ is larger than the average duration of fades of the $E$

field below a level $\tilde{R}_e$ where $\tilde{\Psi}_e = \tilde{R}_e^2 = 1$. The curves of $\mathrm{t}(\tilde{\Psi}_t)$ and $\mathrm{t}(\tilde{R}_e)$ have been plotted in Fig. 4 for comparison.

## V. DISCUSSION OF THE THEORETICAL RESULTS

From the above derivation we know that $\psi_t$ and $\dot{\psi}_t$ are two independent variables as shown by (42):

$$p(\psi_t\,,\,\dot{\psi}_t) = p(\psi_t)p(\dot{\psi}_t),$$

and therefore (5) can be written as follows:

$$\begin{aligned} n(\Psi_t) &= p(\Psi_t)\int_0^\infty \dot{\psi}_t p(\dot{\psi}_t)d\dot{\psi}_t \\ &= p(\Psi_t)\,\{\dot{\psi}_t\} \end{aligned} \tag{58}$$

where $\{\dot{\psi}_t\}$ represents an integral. Equation (58) simply shows that the expected number of crossings $n(\Psi_t)$ at given level $\Psi$ can be obtained from the probability density of level $\Psi_t$ times the integral $\{\dot{\psi}_t\}$. The average duration of fades, then, turns out to be

$$\begin{aligned} \mathrm{t}(\Psi) &= \frac{P(\psi_t < \Psi_t)}{n(\Psi_t)} \\ &= \frac{1}{\{\dot{\psi}_t\}}\frac{P(\psi_t < \Psi_t)}{p(\Psi_t)}. \end{aligned} \tag{59}$$

We emphasize that (58) and (59) are valid only when $\psi_t$ and $\dot{\psi}_t$ are two independent variables.

The two curves, $n(\tilde{R}_e)$ and $n(\tilde{\Psi}_t)$, are plotted in Fig. 3, normalized by the common factor $\sqrt{2\pi}/\beta V$. Both curves are plotted as functions of the signal level normalized to their own rms values. The value of $n(\tilde{R}_e)$ is, as shown, always higher than the value of $n(\tilde{\Psi}_t)$ for any signal level. From these two curves, it may be said that the fading of the energy density is less frequent than the fading of the envelope of the electric field. The maximum expected numbers of crossings of both $n(\tilde{R}_e)$ and $n(\tilde{\Psi}_t)$ are at the $-3$ dB level, which means for signal level at $1/\sqrt{2}$ and $\frac{1}{2}$ of their rms values, respectively, we will count the most fades. The curve of $n(\tilde{\Psi}_t)$ has dropped faster on both sides of 0 dB than the curve of $n(\tilde{R}_e)$, which means that the range of the signal amplitude $\psi_t$ is less than the range of the signal amplitude $r_e$.

The average duration of the signal below a given amplitude level is another way of looking at the fading problem. Fig. 4 shows that the average duration of fades of the energy density $\mathrm{t}(\tilde{\Psi}_t)$ is always larger

than the average duration of fades of the electric field $\mathsf{t}(R_s)$ when the given signal levels are above $-3$ dB with respect to their rms values ($\tilde{\Psi}_s = \tilde{R}_s^2 > -3$ dB). The value of $\mathsf{t}(\tilde{\Psi}_s)$ is less than $\mathsf{t}(\tilde{R}_s)$ when the given signal levels are more than 3 dB below the rms values $\tilde{\Psi}_s = \tilde{R}_s^2 < -3$ dB. When the given signal level is at $-3$ dB ($\tilde{\Psi}_s = \tilde{R}_s^2 = -3$ dB), the average duration of fades $\mathsf{t}(-3$ dB) of both the energy density and the electric field are the same.

## VI. COMPARISON OF THE THEORETICAL PREDICTION WITH THE EXPERIMENTS

The three field components $E$, $H_x$, and $H_y$ have been received by a special antenna[6,7] mounted on a mobile van moving at a speed of 15 mile/hr. All the figures shown in this section were taken on Commonwealth Avenue, New Providence, New Jersey, from a transmitting

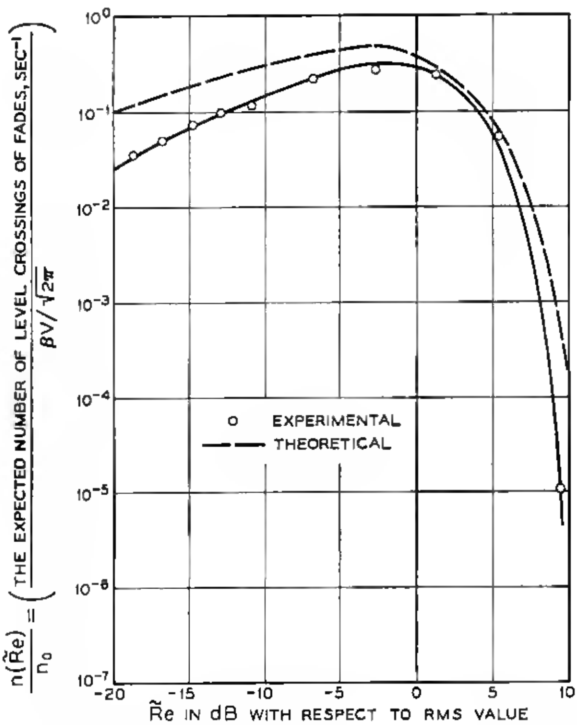


Fig. 6 — Comparison of the predicted level crossing rates to the observed rates for the electric field on Commonwealth Avenue, New Providence, New Jersey.

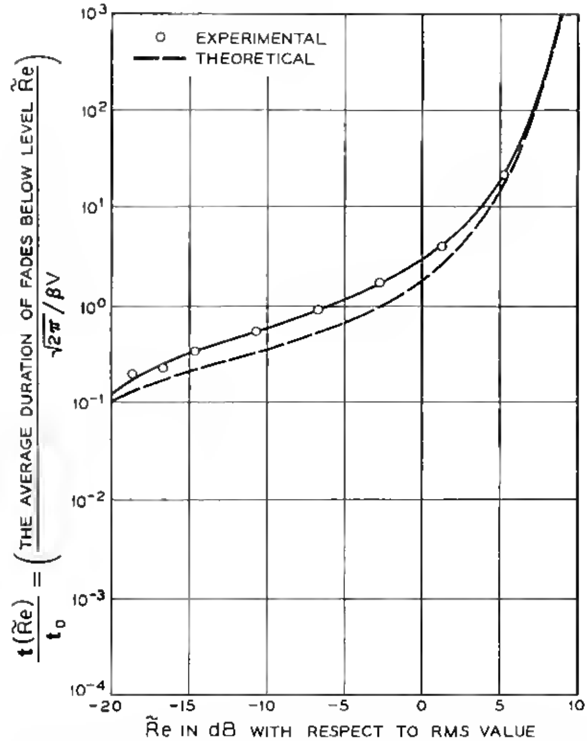


Fig. 7 — Comparison of the predicted average duration of fades to the observed average duration of fades for the electric field on Commonwealth Avenue, New Providence, New Jersey.

antenna at 836 MHz at Bell Laboratories, Murray Hill. After adjusting the appropriate relative gains of the three fields, the energy density can be obtained by squaring and summing these three fields by computer

$$\psi_t = |E|^2 + |H_x|^2 + |H_y|^2, \qquad \text{volt}^2/m^2.$$

Since the distance between the transmitting antenna and the mobile unit is relatively short, the angle swept out by the radius vector from the base station to the mobile unit varies considerably over a typical length of run. To reduce the variation of this angle the data for the entire run were cut into sections 8 seconds long, corresponding to 175 feet of travel, for computer processing. Each section, either the envelope $r_e$ of the $E$ field or the energy density $\psi_t$, was used to obtain the number of level crossings $n$ and the average duration of fades $\mathbf{t}$ by com-

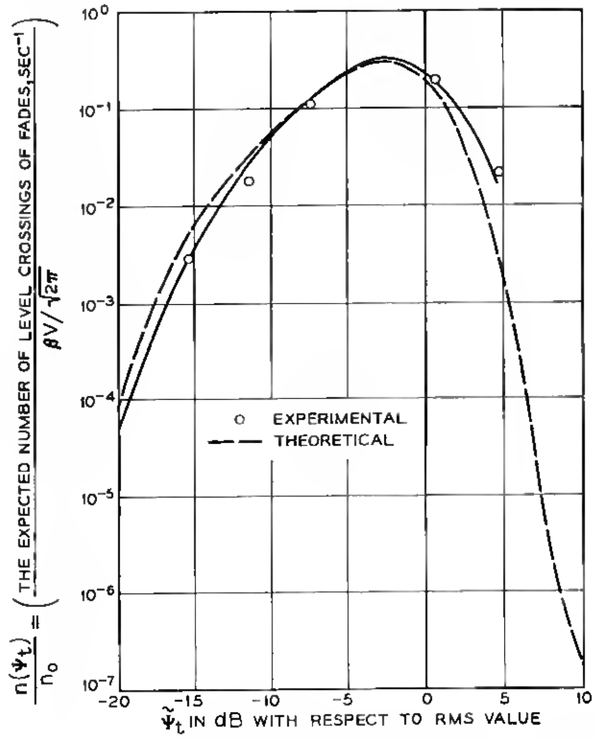


Fig. 8 — Comparison of the predicted level crossing rates to the observed rates for the energy density on Commonwealth Avenue, New Providence, New Jersey.

puter program. However, since the experimental curves of $n$ and $\mathsf{t}$ were almost all alike for all sections, we used only one for comparison with the theoretical curve.

Fig. 6 shows a comparison of the curves of the expected number of crossings $n(\tilde{R}_e)$ at any level $\tilde{R}_e$ for both experiment and theory. The shape of the experimental curve is in fairly good agreement with the theoretical curve. Since the receiving antenna on Commonwealth Avenue is in line of sight with the transmitting antenna at Bell Laboratories, a small direct wave component may be introduced. This small direct wave component is not considered in our theoretical analysis, hence the values $n(\tilde{R}_e)$ from the experiments should be less than the theoretical results as we would predict.

Fig. 7 shows a comparison of the curves of the average duration of fades $\mathsf{t}(\tilde{R}_e)$ for both experiment and theory. They are quite alike. Since

a small direct wave component does exist, the average duration of fades for the experimental data should be higher than the theoretical results.

Fig. 8 shows a comparison of the curves of the expected number of crossings $n(\Psi_t)$ at any level $\tilde{\Psi}_t$ for both experiment and theory. The shape of the experimental curve is very much like the theoretical curve. It shows that the theoretical model used in this paper is quite acceptable.

Fig. 9 shows a comparison of the curves of the average duration of fades $t(\tilde{\Psi}_t)$ for both experiment and theory. The difference between the experimental curve and the theoretical curve may be caused by the small direct wave component. A small direct wave component introduced into our theoretical model may cause a little higher average duration of fades than it might expect, but does not affect the number of level crossings.

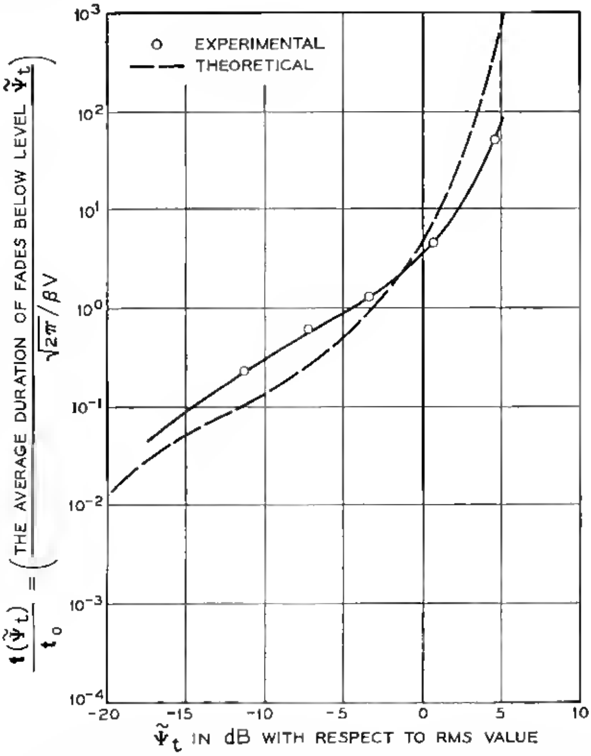


Fig. 9 — Comparison of the predicted average duration of fades to the observed average duration of fades for the energy density on Commonwealth Avenue, New Providence, New Jersey.

## VII. CONCLUSIONS

Comparing the expected number of level crossings and the average durations of fades of the energy density with that of the $E$ field, we see that the fading of the energy density is much less severe than the fading of the envelope of $E$ field.

Referring to Fig. 5, which shows the fading rate related to the orientation of the energy density antenna and the direction of vehicle motion, we see that when the two orthogonal loops are at 45° to the direction of motion, the fades of all three field components are the same. When one loop is lined up with the direction of motion and the other normal to it, the $H$ field component received from the loop normal to the motion has less fading than either of the other two field components.

The expected number of crossings/second of fades at a given signal level, $n$, for both $R_e$ and $\Psi_t$ is proportional to the carrier frequency $f_c$ and the mobile speed $V$, as shown in (26) and (54). They have the common factor, $(\beta V/\sqrt{2\pi} = \sqrt{2\pi}(Vf_c/c)$, where $c$ is the velocity of light. Hence, if either $V$ or $f_c$ goes higher, $n$ becomes greater.

The average duration of fades, $\mathrm{t}$, is inversely proportional to the carrier frequency $f_c$ and mobile speed $V$, as shown in (30) and (57). Hence, if either $V$ or $f_c$ goes higher, $\mathrm{t}$ becomes smaller.

The foregoing theoretical analysis is based on a Gaussian model and does not include a direct wave component. Even so, this analysis is compared with the experimental results in Section VI with fairly good agreement.

## VIII. ACKNOWLEDGMENT

Discussions with W. C. Jakes, Jr. and R. H. Clarke have been very helpful. H. H. Hoffman helped with the experimental data and P. Liu did the computer programming to obtain the statistical results.

## APPENDIX

### A.1 *Finding the mean values, variances and covariances of nine variables* $(X_1, Y_1, X_2, Y_2, X_3, Y_3, \dot{\psi}_e, \dot{\psi}_{hx}, \dot{\psi}_{hy})$.

From (8), (9), and (10) we may express in the following forms

$$E_z = X_1 + jY_1$$

$$H_x = X_2 + jY_2$$

$$H_y = X_3 + jY_3 ,$$

where

$$X_1 = \sum_{u=1}^{N} R_u \cos \Phi_u + S_u \sin \Phi_u \tag{60}$$

$$Y_1 = \sum_{u=1}^{N} S_u \cos \Phi_u - R_u \sin \Phi_u \tag{61}$$

$$X_2 = \sum_{u=1}^{N} (R_u \cos \Phi_u + S_u \sin \Phi_u) \sin \theta_u \tag{62}$$

$$Y_2 = \sum_{u=1}^{N} (S_u \cos \Phi_u - R_u \sin \Phi_u) \sin \theta_u \tag{63}$$

$$X_3 = -\sum_{u=1}^{N} (R_u \cos \Phi_u + S_u \sin \Phi_u) \cos \theta_u \tag{64}$$

$$Y_3 = -\sum_{u=1}^{N} (S_u \cos \Phi_u - R_u \sin \Phi_u) \cos \theta_u \tag{65}$$

also

$$\Phi_u = \beta V t \cos (\theta_u - \alpha) \tag{66}$$

and the angles $\theta_u$ and $\alpha$ are shown in Fig. 2. The time derivatives of $X_1$, $Y_1$, $X_2$, $Y_2$, $X_3$, and $Y_3$ are

$$\dot{X}_1 = \beta V \sum_u (-R_u \sin \Phi_u + S_u \cos \Phi_u) \cos (\theta_u - \alpha) \tag{67}$$

$$\dot{Y}_1 = \beta V \sum_u (-S_u \sin \Phi_u - R_u \cos \Phi_u) \cos (\theta_u - \alpha) \tag{68}$$

$$\dot{X}_2 = \beta V \sum_u (-R_u \sin \Phi_u + S_u \cos \Phi_u) \sin \theta_u \cos (\theta_u - \alpha) \tag{69}$$

$$\dot{Y}_2 = \beta V \sum_u (-S_u \sin \Phi_u - R_u \cos \Phi_u) \sin \theta_u \cos (\theta_u - \alpha) \tag{70}$$

$$\dot{X}_3 = -\beta V \sum_u (-R_u \sin \Phi_u + S_u \cos \Phi_u) \cos \theta_u \cos (\theta_u - \alpha) \tag{71}$$

$$\dot{Y}_3 = -\beta V \sum_u (-S_u \sin \Phi_u - R_u \cos \Phi_u) \cos \theta_u \cos (\theta_u - \alpha). \tag{72}$$

The mean values of all above random variables are zero $\langle X_1 \rangle = \langle Y_1 \rangle = \langle X_2 \rangle = \langle Y_2 \rangle = \langle X_3 \rangle = \langle Y_3 \rangle = \langle \dot{X}_1 \rangle = \langle \dot{Y}_1 \rangle = \langle \dot{X}_2 \rangle = \langle \dot{Y}_2 \rangle = \langle \dot{X}_3 \rangle = \langle \dot{Y}_3 \rangle = 0$. The variances of all above random variables are

$$\left.\begin{aligned}\mu_{11} &= \langle X_1^2\rangle = \langle Y_1^2\rangle = N \\ \mu_{22} &= \langle X_2^2\rangle = \langle Y_2^2\rangle = \frac{N}{2} \\ \mu_{33} &= \langle X_3^2\rangle = \langle Y_3^2\rangle = \frac{N}{2}\end{aligned}\right\} \text{for any } N \qquad \begin{aligned}(73)\\(74)\\(75)\end{aligned}$$

$$\mu_{11}' = \langle \dot{X}_1^2\rangle = \langle \dot{Y}_1^2\rangle = \frac{N}{2}(\beta V)^2 \qquad \text{for } N \geqq 3 \quad (76)$$

$$\left.\begin{aligned}\mu_{22}' &= \langle \dot{X}_2^2\rangle = \langle \dot{Y}_2^2\rangle = \frac{N}{8}(\beta V)^2[\cos^2\alpha + 3\sin^2\alpha] \\ \mu_{33}' &= \langle \dot{X}_3^2\rangle = \langle \dot{Y}_3^2\rangle = \frac{N}{8}(\beta V)^2[3\cos^2\alpha + \sin^2\alpha]\end{aligned}\right\} \text{for} \begin{cases} N = 3 & (77) \\ N \geqq 5. & (78)\end{cases}$$

*Remark*: The values of $\mu_{11}'$ for $N \geqq 3$ is derived as follows: The summations of sine and cosine functions can be expressed

$$\sum_{k=1}^{N} \sin kx = \frac{\cos\frac{x}{2} - \cos(2N+1)\frac{x}{2}}{2\sin\frac{x}{2}}$$

$$\sum_{k=1}^{N} \cos kx = \frac{\sin(2N+1)\frac{x}{2} - \sin\frac{x}{2}}{2\sin\frac{x}{2}}.$$

Then in (67)

$$\theta_u = \frac{2\pi u}{N}$$

and

$$\sum_{u=1}^{N} \cos^2(\theta_u - \alpha) = \frac{1}{2}\sum_{u=1}^{N}\left[1 + \cos\left(u\frac{4\pi}{N} - 2\alpha\right)\right]$$

$$= \frac{N}{2} + \frac{\cos 2\alpha}{2}\sum_{u=1}^{N}\cos u\frac{4\pi}{N} + \frac{\sin 2\alpha}{2}\sum_{u=1}^{N}\sin u\frac{4\pi}{N}.$$

But,

$$\left.\begin{aligned}
&\sum_{u=1}^{N} \cos u \frac{4\pi}{N} = \frac{\sin\left(4\pi + \dfrac{2\pi}{N}\right) - \sin\dfrac{2\pi}{N}}{2 \sin\dfrac{2\pi}{N}} = 0 \\
&\text{and} \\
&\sum_{u=1}^{N} \sin u \frac{4\pi}{N} = \frac{\cos\dfrac{2\pi}{N} - \cos\left(4\pi + \dfrac{2\pi}{N}\right)}{2 \sin\dfrac{2\pi}{N}} = 0
\end{aligned}\right\} \text{for} \quad N \geqq 3.$$

Thus, the average value of $X_1^2$ or $Y_1^2$ is

$$\left\langle \sum_{u=1}^{N} \cos^2\left(\frac{2\pi u}{N} - \alpha\right)\right\rangle_{\text{av}} = \frac{N}{2} \quad \text{for} \quad N \geqq 3.$$

Following the same derivation, we obtain the valid range of $N$ for $\mu_{22}'$ and $\mu_{33}'$. Later on we will obtain the range of $N$ for $\rho_{11}'$, $\rho_{22}'$, $\rho_{33}'$, $\rho_{12}'$, $\rho_{13}'$ and $\rho_{23}'$ in the same way.

*QED.*

Now we are going to find the relations between all the six variables $X_1$, $X_2$, $X_3$, $Y_1$, $Y_2$, $Y_3$ and their time derivatives. Since we know if two variables $a$ and $b$ are Gaussian, and also uncorrelated, $\langle ab\rangle = 0$, then $a$ and $b$ are independent.[16] Therefore, the covariance of $X_1$, $Y_1$, $\dot{X}_1$, and $\dot{Y}_1$ are

$$\langle X_1 Y_1\rangle = \langle X_1 \dot{X}_1\rangle = \langle X_1 \dot{Y}_1\rangle = \langle Y_1 \dot{X}_1\rangle = \langle Y_1 \dot{Y}_1\rangle = \langle \dot{X}_1 \dot{Y}_1\rangle = 0$$

hence, the four variables $X_1$, $Y_1$, $\dot{X}_1$, and $\dot{Y}_1$ are statistically independent. The covariances of $X_2$, $Y_2$, $\dot{X}_2$, and $\dot{Y}_2$ are

$$\langle X_2 Y_2\rangle = \langle X_2 \dot{X}_2\rangle = \langle X_2 \dot{Y}_2\rangle = \langle Y_2 \dot{X}_2\rangle = \langle Y_2 \dot{Y}_2\rangle = \langle \dot{X}_2 \dot{Y}_2\rangle = 0$$

hence, the four variables $X_2$, $Y_2$, $\dot{X}_2$, and $\dot{Y}_2$ are statistically independent. The covariances of $X_3$, $Y_3$, $\dot{X}_3$, and $\dot{Y}_3$ are

$$\langle X_3 Y_3\rangle = \langle X_3 \dot{X}_3\rangle = \langle X_3 \dot{Y}_3\rangle = \langle Y_3 \dot{X}_3\rangle = \langle Y_3 \dot{Y}_3\rangle = \langle \dot{X}_3 \dot{Y}_3\rangle = 0$$

hence, the four variables $X_3$, $Y_3$, $\dot{X}_3$, and $\dot{Y}_3$ are statistically independent.

Also we may show that

$$\langle X_m Y_n\rangle = 0 \qquad \text{for all } m \text{ and } n$$

$$\langle X_m X_n\rangle = \langle Y_m Y_n\rangle = 0 \qquad \text{for } m \neq n,$$

where

$$\left.\begin{matrix} m \\ n \end{matrix}\right\} = 1, 2, 3,$$

hence the six variables $X_1$, $X_2$, $X_3$, $Y_1$, $Y_2$, $Y_3$ are independent. The rate of change of energy densities of three fields $E_z$, $H_x$, and $H_y$ are

$$\begin{aligned} \dot{\psi}_e &= \frac{d}{dt}\psi_e = \frac{d}{dt}(X_1^2 + Y_1^2) \\ &= \beta V \sum_{u,v} [-(R_uR_v + S_uS_v)\sin(\Phi_u - \Phi_v) \\ &\quad + (S_uR_v - R_uS_v)\cos(\Phi_u - \Phi_v)] \times [\cos(\theta_u - \alpha) - \cos(\theta_v - \alpha)], \end{aligned} \tag{79}$$

$$\begin{aligned} \dot{\psi}_{hx} &= \frac{d}{dt}\psi_{hx} = \frac{d}{dt}(X_2^2 + Y_2^2) \\ &= \beta V \sum_{u,v} [-(R_uR_v + S_uS_v)\sin(\Phi_u - \Phi_v) \\ &\qquad + (S_uR_v - R_uS_v)\cos(\Phi_u - \Phi_v)] \\ &\qquad \cdot \sin\theta_u \sin\theta_v[\cos(\theta_u - \alpha) - \cos(\theta_v - \alpha)], \end{aligned} \tag{80}$$

$$\begin{aligned} \dot{\psi}_{hy} &= \frac{d}{dt}\psi_{hy} = \frac{d}{dt}(X_3^2 + Y_3^2) \\ &= \beta V \sum_{u,v} [-(R_uR_v + S_uS_v)\sin(\Phi_u - \Phi_v) \\ &\qquad + (S_uR_v - R_uS_v)\cos(\Phi_u - \Phi_v)] \\ &\qquad \cdot \cos\theta_u \cos\theta_v[\cos(\theta_u - \alpha) - \cos(\theta_v - \alpha)]. \end{aligned} \tag{81}$$

The only terms that exist in (79), (80), (81) are those for which $u \neq v$. There are $N(N - 1)/2$ different terms which are all statistically independent in (79), (80), and (81). Hence, by the central limit theorem, $\dot{\psi}_e$, $\dot{\psi}_{hy}$, and $\dot{\psi}_{hx}$ are Gaussian random variables.

The variance of $\dot{\psi}_e$, $\dot{\psi}_{hx}$ and $\dot{\psi}_{hy}$ are

$$\rho_{11}' = \langle \dot{\psi}_e^2 \rangle = (\beta V)^2\, 4N(N - 1) \qquad \text{for} \quad N \geqq 3 \tag{82}$$

$$\left.\begin{aligned} \rho_{22}' &= \langle \dot{\psi}_{hx}^2 \rangle = (\beta V)^2 \frac{N(N-1)}{2}[\cos^2\alpha + 3\sin^2\alpha] \quad (83) \\ \rho_{33}' &= \langle \dot{\psi}_{hy}^2 \rangle = (\beta V)^2 \frac{N(N-1)}{2}[3\cos^2\alpha + \sin^2\alpha] \quad (84) \end{aligned}\right\} \text{for} \begin{cases} N = 3 \\ N \geqq 5. \end{cases}$$

The covariances of $\psi_e$, $\psi_{hx}$, and $\psi_{hy}$ are

$$\left.\begin{aligned} \rho_{12}' = \rho_{21}' &= \langle \psi_e \psi_{hx} \rangle = -(\beta V)^2 2N(N-1) \sin^2 \alpha \\ \rho_{13}' = \rho_{31}' &= \langle \psi_e \psi_{hy} \rangle = -(\beta V)^2 2N(N-1) \cos^2 \alpha \end{aligned}\right\} \text{for } N \geqq 3 \tag{85, 86}$$

$$\rho_{23}' = \rho_{32}' = \langle \psi_{hx} \psi_{hy} \rangle = 0. \tag{87}$$

It is very easy to show from (60) to (66) and (79) to (81) that the covariances of the variables between two groups ($\psi_e$, $\psi_{hx}$, $\psi_{hy}$) and ($X_1$, $Y_1$, $X_2$, $Y_2$, $X_3$, $Y_3$) are zero. We may write

$$\langle \psi_m X_n \rangle = \langle \psi_m Y_n \rangle = 0 \qquad \text{for all } n \text{ and } m \tag{88}$$

$$\left.\begin{matrix} m \\ n \end{matrix}\right\} = 1, 2, 3$$

hence ($\psi_e$, $\psi_{hx}$, $\psi_{hy}$) and ($X_1$, $Y_1$, $X_2$, $Y_2$, $X_3$, $Y_3$) are two independent variable groups.[16]

### A.2 *Derivation of* $M_{\psi_t}(jv)$ *in* (50)

The mean values of all three Gaussian random variables $\psi_e$, $\psi_{hx}$, and $\psi_{hy}$ we observed from (79) to (81) are zero. Also (87) gives the covariance $\langle \psi_{hx} \psi_{hy} \rangle = 0$. The joint probability density function of three variables $\psi_e$, $\psi_{hx}$, $\psi_{hy}$ can be obtained[11]

$$\begin{aligned} p(\psi_e, \psi_{hx}, \psi_{hy}) = \frac{1}{(2\pi)^{\frac{3}{2}} \mid \Lambda \mid^{\frac{1}{2}}} \exp \biggl\{ -\frac{1}{2 \mid \Lambda \mid} (\mid \Lambda_{11} \mid \psi_e^2 + \mid \Lambda_{22} \mid \psi_{hx} \\ + \mid \Lambda_{33} \mid \psi_{hy}^2 + 2 \mid \Lambda_{12} \mid \psi_e \psi_{hx} + 2 \mid \Lambda_{13} \mid \psi_e \psi_{hy} \\ + 2 \mid \Lambda_{23} \mid \psi_{hx} \psi_{hy}) \biggr\}, \end{aligned}$$

where

[Λ] is a covariance matrix, and
$|\Lambda_{mn}|$ is a cofactor of $\rho_{mn}$ in the covariance matrix [Λ]

$$[\Lambda] = \begin{bmatrix} \rho_{11}' & \rho_{12}' & \rho_{13}' \\ \rho_{12}' & \rho_{22}' & 0 \\ \rho_{13}' & 0 & \rho_{33}' \end{bmatrix} \tag{89}$$

$$|\Lambda| = \text{determinant of } [\Lambda] = \rho'_{33}(\rho'_{11}\rho'_{22} - \rho'^2_{12}) - \rho'_{22}\rho'^{2}_{13}$$

$$| \Lambda_{11} | = \rho'_{22}\rho'_{33}$$

$$| \Lambda_{22} | = \rho'_{11}\rho'_{33} - \rho'^{2}_{13}$$

$$| \Lambda_{33} | = \rho'_{11}\rho'_{22} - \rho'^{2}_{12}$$

$$| \Lambda_{12} | = | \Lambda_{21} | = -\rho'_{12}\rho'_{33}$$

$$| \Lambda_{13} | = | \Lambda_{31} | = -\rho'_{22}\rho'_{13}$$

$$| \Lambda_{23} | = | \Lambda_{32} | = \rho'_{13}\rho'_{12} .$$

From (38), $\dot{\psi}_t$ is the sum of three Gaussian random variables $\dot{\psi}_e$ , $\dot{\psi}_{hx}$ , $\dot{\psi}_{hy}$ :

$$\dot{\psi}_t = \dot{\psi}_e + \dot{\psi}_{hx} + \dot{\psi}_{hy} .$$

The characteristic function for $\dot{\psi}_t$ is then

$$M_{\dot{\psi}_t}(jv) = E[\exp \{jv(\dot{\psi}_e + \dot{\psi}_{hx} + \dot{\psi}_{hy}\}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp \{jv(\dot{\psi}_e + \dot{\psi}_{hx} + \dot{\psi}_{hy})\}p(\dot{\psi}_e , \dot{\psi}_{hx} , \dot{\psi}_{hy})d\dot{\psi}_e d\dot{\psi}_{hx} d\dot{\psi}_{hy}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}} | \Lambda |^{\frac{1}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{jv(\dot{\psi}_e+\dot{\psi}_{hx})}$$

$$\cdot \exp \left\{-\frac{1}{2 | \Lambda |} (| \Lambda_{11} | \dot{\psi}_e^2 + | \Lambda_{22} | \dot{\psi}_{hx}^2 + 2 | \Lambda_{12} | \dot{\psi}_e\dot{\psi}_{hx})\right\}$$

$$\times \int_{-\infty}^{\infty} e^{jv\dot{\psi}_{hy}} \exp \left\{-\frac{| \Lambda_{33} |}{2 | \Lambda |} \left(\dot{\psi}_{hy}^2 + \frac{2(| \Lambda_{13} | \dot{\psi}_e + | \Lambda_{23} | \dot{\psi}_{hx})}{| \Lambda_{33} |} \dot{\psi}_{hy}\right)\right\}$$

$$\cdot d\dot{\psi}_{hy} d\dot{\psi}_{hx} d\dot{\psi}_e . \tag{90}$$

The last integrand of $\dot{\psi}_y$ is

$$\exp \left\{-j \frac{| \Lambda_{13} | \dot{\psi}_e + | \Lambda_{23} | \dot{\psi}_{hx}}{| \Lambda_{33} |} v + \frac{| \Lambda_{33} |}{2 | \Lambda |} \left(\frac{| \Lambda_{13} | \dot{\psi}_e + | \Lambda_{23} | \dot{\psi}_{hx}}{| \Lambda_{33} |}\right)^2\right\}$$

$$\times \int_{-\infty}^{\infty} \exp \left[-jv\left(\dot{\psi}_{hy} + \frac{| \Lambda_{13} | \dot{\psi}_e + | \Lambda_{23} | \dot{\psi}_{hx}}{| \Lambda_{33} |}\right)\right]$$

$$\exp \left\{-\frac{| \Lambda_{33} |}{2 | \Lambda |} \left(\dot{\psi}_y + \frac{| \Lambda_{13} | \dot{\psi}_e + | \Lambda_{23} | \dot{\psi}_{hx}}{| \Lambda_{33} |}\right)^2\right\} d\dot{\psi}_{hy}$$

$$= \exp \left(-jv\left(\frac{| \Lambda_{13} | \dot{\psi}_e + | \Lambda_{23} | \dot{\psi}_{hx}}{| \Lambda_{33} |}\right)\right.$$

$$\left. + \frac{| \Lambda_{33} |}{2 | \Lambda |} \left(\frac{| \Lambda_{13} | \dot{\psi}_e + | \Lambda_{23} | \dot{\psi}_{hx}}{| \Lambda_{33} |}\right)^2\right\}$$

$$\times \int_{-\infty}^{\infty} \exp \left(jv\xi - \frac{h_1}{2} \xi^2\right) d\xi.$$

From Cramer,[15] p. 99 we obtain

$$\int_{-\infty}^{\infty} \exp\left(jv\xi - \frac{h_1}{2}\,\xi^2\right)d\xi = \sqrt{\frac{2\pi}{h_1}}\,\exp -\frac{v^2}{2h_1}\,,$$

where

$$h_1 = \frac{|\,\Lambda_{33}\,|}{|\,\Lambda\,|}.$$

Then following the same techniques we find

$$\begin{aligned}
M_{\psi_t}(jv) &= \frac{1}{(2\pi)^{\frac{3}{2}}\,|\,\Lambda\,|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left(jv_3\xi_3 - \frac{h_3}{2}\,\xi_3^2\right)d\xi_3 \\
&\quad\cdot \int_{-\infty}^{\infty} \exp\left(jv_2\xi_2 - \frac{h_2}{2}\,\xi_2^2\right)d\xi_2 \int_{-\infty}^{\infty} \exp\left(jv_1\xi_1 - \frac{h_1}{2}\,\xi_1^2\right)d\xi_1 \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}\,|\,\Lambda\,|^{\frac{1}{2}}} \sqrt{\frac{(2\pi)^3}{h_1h_2h_3}} \left\{\exp\left[-\frac{1}{2}\left(\frac{v_1^2}{h_1} + \frac{v_2^2}{h_2} + \frac{v_3^2}{h_3}\right)\right]\right\}, \qquad (91)
\end{aligned}$$

where

$$\begin{aligned}
h_1 &= \frac{|\,\Lambda_{33}\,|}{|\,\Lambda\,|} \\
h_2 &= \frac{B}{|\,\Lambda\,|\,|\,\Lambda_{33}\,|} \\
h_3 &= \frac{1}{|\,\Lambda\,|\,|\,\Lambda_{33}\,|\,B}\,[BC - A^2] \\
v_1 &= v \\
v_2 &= \left(1 - \frac{|\,\Lambda_{23}\,|}{|\,\Lambda_{33}\,|}\right)v \\
v_3 &= \left[1 - \frac{|\,\Lambda_{13}\,|}{|\,\Lambda_{33}\,|} - \left(1 - \frac{|\,\Lambda_{23}\,|}{|\,\Lambda_{33}\,|}\right)\frac{A}{B}\right]v \\
A &= |\,\Lambda_{12}\,|\,|\,\Lambda_{33}\,| - |\,\Lambda_{13}\,|\,|\,\Lambda_{23}\,| = -\rho_{12}'\,|\,\Lambda\,| \\
B &= |\,\Lambda_{22}\,|\,|\,\Lambda_{33}\,| - |\,\Lambda_{23}\,|^2 = \rho_{11}'\,|\,\Lambda\,| \\
C &= |\,\Lambda_{11}\,|\,|\,\Lambda_{33}\,| - |\,\Lambda_{13}\,|^2 = \rho_{22}'\,|\,\Lambda\,| \\
BC - A^2 &= |\,\Lambda_{33}\,|\,|\,\Lambda\,|^2.
\end{aligned}$$

The constant value outside the bracket of (91) is

$$\frac{1}{(2\pi)^{\frac{3}{2}}\,|\,\Lambda\,|^{\frac{1}{2}}} \sqrt{\frac{(2\pi)^3}{h_1h_2h_3}} = \frac{1}{|\,\Lambda\,|^{\frac{1}{2}}} \sqrt{\frac{|\,\Lambda\,|^3\,|\,\Lambda_{33}\,|}{BC - A^2}} = 1$$

and the expression inside the bracket of (91) is

$$\exp -\frac{1}{2}\left[\frac{v_1^2}{h_1}+\frac{v_2^2}{h_2}+\frac{v_3^2}{h_3}\right]$$

$$= \exp\left\{-\frac{1}{2}\frac{|\Lambda|}{|\Lambda_{33}|}\left[1+\frac{(|\Lambda_{33}|-|\Lambda_{23}|)^2}{B}\right.\right.$$

$$\left.\left.+\frac{\left[|\Lambda_{33}|-|\Lambda_{13}|-(|\Lambda_{33}|-|\Lambda_{23}|)\frac{A}{B}\right]^2 B}{BC-A^2}\right]v^2\right\}$$

$$= \exp\{-\tfrac{1}{2}(\rho_{11}'+\rho_{22}'+\rho_{33}'+2\rho_{12}'+2\rho_{13}')v^2\} = \exp -\tfrac{1}{2}\rho_t' v^2.$$

Thus,

$$M_{\dot{\psi}_t}(jv) = \exp -\tfrac{1}{2}\rho_t' v^2. \tag{92}$$

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